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Brane Cosmology Solutions with Bulk Scalar Fields

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Abstract: Brane cosmologies with static, five-dimensional and Z_2 symmetric bulks are analysed. A general solution generating mechanism is outlined. The qualatitive cosmological behaviour of all such solutions is determined. Conditions for avoiding naked bulk singularities are also discussed. The restrictions placed on the solutions by the assumption of such a static bulk are investigated. In particular the requirement of a non-standard energy-momentum conservation law. The failure of such solutions to provide viable quintessence terms in the Friedmann equations is also shown.

Keywords: Extra Large Dimensions, Physics of the Early Universe, Cosmology of Theories beyond the SM.

1. Introduction

The possibility that our four-dimensional universe may be embedded in a higher dimensional 'bulk' spacetime has received a great deal of attention from cosmologists. This idea is motivated by the existence of D-branes in string/M-theory. Our universe could then be one of these branes, moving through a non-compact higher dimensional space. In general, effects arising from the extra dimensions in this setup will be incompatible with the observed behaviour of our universe. However, if a negative cosmological constant is added to the bulk, it is possible (to leading order) to re-obtain standard four-dimensional gravity [1] and cosmology on the brane [2].

Early work concentrated on models with constant bulks, however in the context of string/M-theory it is natural for bulk scalar fields to be present too, and so their effects should not be neglected. It is also possible that the variation of these fields could provide new mechanisms for previously unexplained cosmological phenomena. For example, they may produce effective quintessence terms in the brane Friedmann equation. This could then account for the observed acceleration of our universe's expansion.

By projecting the higher dimensional curvature tensors onto the brane, and applying the relevant boundary conditions it is possible to obtain four-dimensional Einstein equations [3]. In the case of brane cosmologies with constant bulk energy densities, this approach allows the evolution of the fields on the brane to be completely determined. When scalar fields are introduced, the behaviour of the projected Weyl tensor is no longer fully determined by the boundary conditions. We therefore need more information about the bulk before we can find the effective Einstein equations on the brane.

One possibility is to to make assumptions about the bulk solution near the brane [4, 5], although we will not know if they are justified or not. A preferable option is to solve the full higher dimensional Einstein equations. So far no general solutions have been found. However solutions can be found in a few special cases in which the equations simplify. For example, we could assume the bulk energy density has the form of a perfect fluid [6]. Alternatively, if the bulk potential energy resembles that of a supergravity theory, BPS solutions can be found [7].

In this paper I will consider models with just one scalar field and a single 3-brane in a five-dimensional bulk. I will be mainly concerned with Z_2 symmetric spacetimes (i.e. the solution on one side of the brane is a reflection of that on the other), although a few results apply to the more general case. To simplify the analysis I will only consider solutions with static bulks. The position of the brane will still be time-dependent, and it is from the brane's movement through the bulk that the cosmological evolution of our universe arises. This generalises an idea by Ida [8]. It turns out that the BPS solutions considered elsewhere are a subset of those considered in this paper.

In section 2 of this paper I will outline the model used, and relate the brane Friedmann equation to the bulk solutions. The assumption of having a static bulk places restrictions on the brane's evolution, which I will discuss. In section 3 I will show

how the static assumption allows the field equations to be simplified, and how they provide a way of generating solutions. The qualitative cosmological properties of all such solutions are discussed in sections 4 and 5. Section 5 deals with a special case of conformally flat solutions, which have been more widely discussed in the literature [9]. For illustration, a particular set of solutions is presented in section 6. Finally the results are summarised in section 7.

2. Static Bulk Brane World Cosmology

For a 3-brane in a 5-dimensional spacetime, the 4-dimensional effective Einstein tensor $\widehat{G}_{\mu\nu}$ is related to a projection of the bulk Einstein tensor [3].

$$\widehat{G}_{\mu\nu} = \frac{2}{3}\left(G_{\lambda\sigma}\hat{g}^{\lambda}_{\mu}\hat{g}^{\sigma}_{\nu} + \left(G_{\lambda\sigma}n^{\lambda}n^{\sigma} - \frac{1}{4}G^{\sigma}{}_{\sigma}\right)\hat{g}_{\mu\nu}\right) \\ -\frac{1}{2}\left(K_{\mu\lambda} - K\hat{g}_{\mu\lambda}\right)\left(K^{\lambda}_{\nu} - K\hat{g}^{\lambda}_{\nu}\right) - E_{\mu\nu}\ , \tag{2.1}$$

where

$$K_{\mu\nu} = \hat{g}^{\lambda}_{\mu}\hat{g}^{\sigma}_{\nu}\nabla_{(\lambda}n_{\sigma)} \tag{2.2}$$

is the extrinsic curvature of the brane,

$$\hat{g}_{\mu\nu} = g_{\mu\nu} - n_{\mu}n_{\nu} \tag{2.3}$$

is the induced metric,

$$E_{\mu\nu} = C^{\alpha}{}_{\lambda\beta\sigma}n_{\alpha}n^{\beta}\hat{g}^{\lambda}_{\mu}\hat{g}^{\sigma}_{\nu} \tag{2.4}$$

is the electric part of the Weyl tensor, and n^{μ} is the brane's normal.

Both $G_{\mu\nu}$ and $K_{\mu\nu}$ are directly determined by the theory's energy momentum tensor. When scalar fields are present, $E_{\mu\nu}$ is not. Thus a full bulk solution is required.

Rather than look for a completely general solution, I will consider the simpler problem of finding solutions with static bulks. The cosmological evolution of the system then arises from the movement of the brane through the bulk. I will also assume that the solution on the brane is homogeneous and isotropic, as in the standard cosmology. In this case the bulk metric can be written as

$$ds^2 = -f^2(r)h(r)dT^2 + r^2\Omega_{ij}dx^i dx^j + \frac{dr^2}{h(r)} \tag{2.5}$$

where Ω_{ij} is the three dimensional metric of space with constant curvature $k = -1, 0, 1$.

A suitable action for a brane world with one bulk scalar field is

$$S = \int_{\text{bulk}} d^5x\sqrt{-g}\left\{\frac{1}{2\kappa^2}R - \frac{1}{2}(\nabla\phi)^2 - V(\phi)\right\} - \int_{\text{brane}} d^4x\sqrt{-\hat{g}}\left\{\frac{4}{\kappa^2}\langle K\rangle + \mathcal{L}_{\text{b}}(\phi)\right\}\ . \tag{2.6}$$

The brane Lagrangian $\mathcal{L}_{\text{b}}$ includes all the Standard model fields, which are confined to the brane, and any brane contributions to the potential of the bulk scalar field ϕ.

The notation $[X] = X^+ - X^-$, $\langle X\rangle = (X^+ + X^-)/2$, which denotes the change and average of quantities across the brane, has been introduced. $\kappa^2 = 8\pi/M_5^3$, where M_5 is the fundamental five-dimensional Planck mass.

The energy-momentum tensor of the bulk scalar field is

$$T^\mu_\nu = \left(\nabla^\mu\phi\nabla_\nu\phi - \delta^\mu_\nu\{\tfrac{1}{2}(\nabla\phi)^2 + V(\phi)\}\right) \ . \tag{2.7}$$

Since we are considering a static bulk, $\phi = \phi(r)$. Two components of the Einstein equation are then

$$G^T_T = \frac{3}{r^2}\left(h + \frac{rh'}{2} - k\right) = -\kappa^2\left(\frac{1}{2}h\phi'^2 + V\right) \tag{2.8}$$

$$G^r_r = G^T_T + \frac{3hf'}{rf} = \kappa^2\left(\frac{1}{2}h\phi'^2 - V\right) \ , \tag{2.9}$$

and the equation of motion of the scalar field is

$$\left\{\frac{3h}{r} + \frac{hf'}{f} + h'\right\}\phi' + h\phi'' = \frac{dV}{d\phi} \ . \tag{2.10}$$

The remaining bulk Einstein equations need not be considered since the Bianchi identities ensure that they are automatically satisfied. Subtracting eq. (2.8) from eq. (2.9) we see that

$$f = \exp\left(\frac{\kappa^2}{3}\int r\phi'^2 dr\right) \ . \tag{2.11}$$

I will now consider the boundary conditions of the bulk solution, which arise from the presence of the brane. Its position is given by $r = a(t)$, $T = T_\mathrm{b}(t)$, where t is the cosmological time experienced on the brane. The induced metric is then

$$ds^2_\mathrm{b} = -dt^2 + a^2(t)\Omega_{ij}dx^i dx^j \ . \tag{2.12}$$

This is a generalisation of the solutions of Ida [8].

The brane's tangent vector is $u^\mu = (\dot{T}, 0, 0, 0, \dot{r})$, where dots denote differentiation by t. This gives the normalisation condition

$$-f^2h^2\dot{T}^2 + \dot{r}^2 = -h \ . \tag{2.13}$$

The unit 1-form normal to the brane is then $n_\mu = (-\dot{r}, 0, 0, 0, \dot{T})f$. I will take $\dot{T}_\mathrm{b}$ and f to be positive, so that t and T have the same direction, and n_μ is an outward normal.

The non-vanishing components of $K_{\mu\nu}$ are

$$K_{\mu\nu}u^\mu u^\nu = -\frac{\ddot{r} + h'/2}{fh\dot{T}} - (\ln f)' fh\dot{T} \tag{2.14}$$

$$K^i{}_j = \frac{fh\dot{T}}{r}\delta^i_j \tag{2.15}$$

For the metrics such as (2.5), the spatial components of $E_{\mu\nu}$ (2.4) satisfy

$$E^i{}_j = \frac{1}{6}\left(\frac{1}{3}G^i{}_i - G^0{}_0 - G^5{}_5\right)\delta^i_j + \frac{h-k}{r^2}\delta^i_j \ . \tag{2.16}$$

The only other non-zero component of the projected Weyl tensor is $E_{\mu\nu}u^\mu u^\nu = E^i{}_i$.

The action (2.6) implies the following junction conditions

$$2\left\langle n^\mu \nabla_\mu \phi \right\rangle = \frac{\delta \mathcal{L}_{\rm b}}{\delta \phi} \tag{2.17}$$

$$2\left\langle K_{\mu\nu} - K\hat{g}_{\mu\nu} \right\rangle = -\kappa^2 S_{\mu\nu} \tag{2.18}$$

on the brane, where $S_{\mu\nu}$ is the energy momentum tensor corresponding to $\mathcal{L}_{\rm b}$.

For a perfect fluid $S^{\mu\nu} = (\rho_{\rm b} + p_{\rm b})u^\mu u^\nu + p_{\rm b}\hat{g}^{\mu\nu}$. The spatial part of the junction condition (2.18) then implies

$$\left\langle fh\dot{T} \right\rangle = -\frac{\kappa^2}{6}a\rho_{\rm b} \ . \tag{2.19}$$

By considering the change in the normalisation condition (2.13) across the brane we find

$$\left[fh\dot{T}\right] = -\frac{3[h]}{\kappa^2 a \rho_{\rm b}} \ . \tag{2.20}$$

The average of eq. (2.13) gives the Friedmann equation on the brane

$$\left(\frac{\dot{a}}{a}\right)^2 = -\frac{k}{a^2} + \langle U \rangle + \frac{\kappa^4}{36}\rho_{\rm b}^2 + \frac{9[U]^2}{4\kappa^4\rho_{\rm b}^2} \ , \tag{2.21}$$

where I have introduced

$$U = -\frac{h-k}{r^2} \ . \tag{2.22}$$

I will now examine the conservation of energy-momentum on the brane. The Codacci equation relates a projection of the bulk Einstein tensor to the extrinsic curvature

$$n^\mu G_{\mu\nu}\hat{g}^\nu_\lambda = D_\mu K^\mu_\lambda - D_\lambda K \ . \tag{2.23}$$

Combining this with $G_{\mu\nu} = \kappa^2 T_{\mu\nu}$ and the junction condition (2.18) gives the non-standard energy momentum conservation equation on the brane

$$\dot{\rho}_{\rm b} + 3\frac{\dot{a}}{a}(\rho_{\rm b} + p_{\rm b}) = -2\dot{a}\left\langle fh\dot{T}\phi'^2 \right\rangle \ . \tag{2.24}$$

The ϕ junction condition (2.17) implies

$$2\left\langle fh\dot{T}\phi' \right\rangle = \frac{\delta\mathcal{L}_{\rm b}}{\delta\phi} \ . \tag{2.25}$$

If we assume a Z_2 symmetric bulk, the above two equations simplify. With the help of the relation between ϕ' and f (2.11), the energy conservation equation becomes

$$\dot{\rho}_{\rm b} + 3\frac{\dot{a}}{a}(\rho_{\rm b} + p_{\rm b}) = \frac{\delta\mathcal{L}_{\rm b}}{\delta\phi}\dot{\phi} = \left(-\frac{1}{f}\frac{df}{d\phi}\rho_{\rm b}\right)\dot{\phi} \ . \tag{2.26}$$

As with the usual brane world cosmology, the Friedmann equation (2.21) has a $\rho_{\rm b}^2$ rather than a $\rho_{\rm b}$ term. To obtain agreement with the standard cosmology at late times, $\rho_{\rm b}$ needs to be composed into a brane tension part and a ordinary energy density part. In contrast to the usual brane cosmology, we also need to satisfy the non-standard energy conservation equation. The simplest way to do this is to take

$$\rho_{\rm b} = (\lambda + \rho)/f \tag{2.27}$$

and similarly $p_{\rm b} = (-\lambda + p)/f$. The energy density ρ corresponds to the normal matter fields on the brane and obeys the usual energy momentum conservation. The Friedmann equation then becomes

$$\left(\frac{\dot{a}}{a}\right)^2 = -\frac{k}{a^2} + U + \frac{\kappa^4\lambda^2}{36f^2} + \frac{\kappa^4}{36f^2}\rho^2 + \frac{\kappa^4\lambda}{18f^2}\rho \ . \tag{2.28}$$

We see that the effective four-dimensional Einstein constant is

$$8\pi G = \frac{\kappa^4\lambda}{6f^2(a)} \ , \tag{2.29}$$

which is generally time dependent. If we require a vanishing effective four-dimensional cosmological constant at late times U then must tend to $-\kappa^4/(36f^2)$.

An alternative solution to eq. (2.26) is to take $\rho_{\rm b} = V_{\rm b}(\phi) + \rho$, where ρ obeys the standard conservation equation (or any other desired law). The function $V_{\rm b}(\phi)$ must then be chosen to satisfy eq. (2.26). Unfortunately this will require a brane potential which changes form as the universe changes from radiation to matter domination, so this type of solution is rather contrived.

In order for a solution of the form (2.27) to be possible, the brane matter fields must have ϕ-dependent couplings in the Lagrangian. There are various ways to do this. One possibility (i) is to assume that brane matter is coupled to a metric which is conformally related to $\hat{g}_{\mu\nu}$ [4], so $\mathcal{L}_{\rm b} = \mathcal{L}_{\rm b}(\beta(\phi)g_{\mu\nu}, \ldots)$ where '$\ldots$' represents all the matter fields which are confined to the brane. Alternatively we could use a Lagrangian of the form $\mathcal{L}_{\rm b} = \beta(\phi)\mathcal{L}_0(g_{\mu\nu}, \ldots)$, $\mathcal{L}_0$ being ϕ independent. Exactly what the on-shell value of $\mathcal{L}_{\rm b}$ is depends on the underlying theory. Two such possibilities are (ii) $\mathcal{L}_{\rm b} = -2p_{\rm b}$ [5] and (iii) $\mathcal{L}_{\rm b} = 2\rho_{\rm b}$ [10]. These three types of ϕ-dependent Lagrangian have

$$\frac{\delta\mathcal{L}_{\rm b}}{\delta\phi} = \frac{2}{\beta}\frac{d\beta}{d\phi}\begin{cases} (\rho_{\rm b} - 3p_{\rm b})/4 & \text{(i)} \\ -p_{\rm b} & \text{(ii)} \\ \rho_{\rm b} & \text{(iii)} \end{cases} \tag{2.30}$$

For a cosmological constant ($p = -\rho$) these all imply $\beta \propto f^{-1/2}$. More generally when $p = w\rho$, β needs to be proportional to (i) $f^{2/(3w-1)}$, (ii) $f^{-1/(2w)}$, or (iii) $f^{-1/2}$. Thus cases (i) and (ii) require that energy densities with different equations of state couple to ϕ differently. In addition, the above setup does not work in case (i) for radiation ($w = 1/3$) or in case (ii) for matter ($w = 0$). Similar restrictions on the form of $\mathcal{L}_{\rm b}$ are discussed in ref. [11].

3. Solving the Field Equations

It is convenient to define

$$\frac{\mathcal{C}}{r^4} = -\frac{1}{3}E^i{}_i = \frac{r}{4f^2}\left(\frac{hf^2}{r^2}\right)' + \frac{k}{2r^2} \tag{3.1}$$

where the Einstein equations have been used to simplify the expression (2.16).

Using the quantities $\mathcal{C}$ (3.1) and U (2.22), the field equations (2.8–2.10) can be rearranged to give

$$\mathcal{C}' = -\frac{\kappa^2}{3}\left(\mathcal{C} - \frac{kr^2}{2}\right) r\phi'^2 \tag{3.2}$$

$$U' + 4\frac{\mathcal{C}}{r^5} = -\frac{2\kappa^2}{3}\left(U - \frac{k}{r^2}\right) r\phi'^2 \tag{3.3}$$

$$\kappa^2 V = 6U + \frac{3}{4}rU' - 3\frac{\mathcal{C}}{r^4} \; . \tag{3.4}$$

If we take ϕ to be a constant we obtain the usual brane cosmology solutions, with $f = 1$, $\mathcal{C}$ constant, and

$$U = -\frac{h-k}{r^2} = -\ell^2 + \frac{\mathcal{C}}{r^4} \; , \tag{3.5}$$

where ℓ is a constant. The Friedmann equation (2.21) then agrees with that derived elsewhere [8, 12].

For a given choice of $r\phi'$ or $\mathcal{C}$, eqs. (3.2) and (3.3) can be integrated. For example if we take $\kappa r\phi' = \alpha\sqrt{3}$, with α constant,

$$\mathcal{C} = \frac{kr^2}{2(2+\alpha^2)} + A_1 r^{-\alpha^2} \tag{3.6}$$

$$U = -\frac{\alpha^2(1+\alpha^2)}{(2+\alpha^2)(1-\alpha^2)}\frac{k}{r^2} + A_1\frac{4}{4-\alpha^2}r^{-4-\alpha^2} + A_2 r^{2\alpha^2} \; . \tag{3.7}$$

This extends the solution presented in refs. [11, 13].

If $k = 0$, the equations (3.2–3.4) simplify to

$$-\frac{\kappa^2}{3}r\phi' = \frac{1}{\mathcal{C}}\frac{d\mathcal{C}}{d\phi} \tag{3.8}$$

$$d\left(\frac{U}{\mathcal{C}^2}\right) = \frac{1}{\mathcal{C}}d(r^{-4}) \tag{3.9}$$

$$V = -\frac{9}{2\kappa^4}\frac{d\mathcal{C}}{d\phi}\frac{d}{d\phi}\left(\frac{U}{\mathcal{C}}\right) + \frac{6}{\kappa^2}U \; . \tag{3.10}$$

The expression for f (2.11) then reduces to $f \propto 1/\mathcal{C}$, and so the Friedmann equation (2.28) becomes

$$\left(\frac{\dot{a}}{a}\right)^2 = U + \frac{\kappa^4}{36}\left(\lambda^2 + \rho^2 + 2\lambda\rho\right)\mathcal{C}^2 \; . \tag{3.11}$$

Thus, given $\mathcal{C}(\phi)$, it is possible to determine the Friedmann equation (from U), the potential V as an explicit function of ϕ, and the evolution of ϕ with respect to r. Ideally we would like to find all possible U and $\mathcal{C}$ from an arbitrary V. Unfortunately the non-linearity of the above equations makes this practically impossible, although general solutions have been found for exponential potentials [14].

The above results can also be derived using coordinates in which the position of the brane is fixed. Following ref. [2] we can take a metric of the form

$$ds^2 = -N^2(t,y)dt^2 + A^2(t,y)\Omega_{ij}dx^i dx^j + dy^2 \; . \tag{3.12}$$

The brane is fixed at $y = 0$. If we then make the assumption that $\phi = \phi(A)$, and define h and f as

$$(\partial_y A)^2 - \left(\frac{\partial_t A}{N}\right)^2 = h(A) \tag{3.13}$$

$$N \propto (\partial_t A) f(A) \; , \tag{3.14}$$

the results of this paper will be re-obtained, with $r = A$. Using this approach some of he results of section 5 were found in ref. [15]. Other related work appears in ref. [16].

4. General Properties of Static Bulk Solutions

Using eq. (3.8) I will now determine the qualitative features of all $k = 0$ brane world cosmologies with static bulks (except for the special case $\mathcal{C} \equiv 0$, which is covered in the next section). In general, ϕ simply rolls in the direction of decreasing $|\mathcal{C}|$ as r increases. More interesting things happen when $d\mathcal{C}/d\phi = 0$, $\mathcal{C} = 0$ or $|\mathcal{C}| \to \infty$. The behaviour of the solutions near these points is easily seen using

$$\ln r = -\frac{\kappa^2}{3} \int \frac{\mathcal{C} d\phi}{(d\mathcal{C}/d\phi)} + \text{const.} \tag{4.1}$$

which is obtained from eq. (3.8). If $d\mathcal{C}/d\phi \to 0$ then the integral diverges and so $\ln r \to \pm\infty$, depending on the sign of the integrand. Thus $r \to 0$ or ∞. If $\mathcal{C}$ passes through zero then $dr/d\phi$ will change sign. In other words the universe will stop expanding and start to contract. If $\mathcal{C}$ diverges then $\ln r$ will either tend to a constant or $-\infty$ depending on how $\mathcal{C}$ diverges. The other possibility ($\ln r \to +\infty$) does not occur, since eq. (3.2) with $k = 0$ implies $d(\ln r)/d(\ln |\mathcal{C}|) \leq 0$, and so r must decrease as $|\mathcal{C}| \to \infty$.

It should be noted that the qualitative cosmological behaviour of the above solution is not completely determined by eqs. (3.8) and (3.9). We also need to consider the possibility of the U term in the Friedmann equation (3.11) cancelling the other terms, to give $\dot{a} = 0$. In general this could occur at any value of $\mathcal{C}$ or $d\mathcal{C}/d\phi$. Depending on the forms of U, $\mathcal{C}$ and ρ it could take an infinite amount of cosmological time to reach this point, or it could be reached in finite time, after which the universe would start to collapse. The subsequent evolution of the universe is then simply the reverse of that up to the point where $\dot{a} = 0$. This type of evolution will prevent some values of ϕ being reached by certain solutions.

We are mainly interested in cosmological solutions which start at $r = 0$ and end at $r = \infty$. The above analysis suggests we want solutions which start at either a suitable turning point or an infinity of $\mathcal{C}$. They should then have ϕ rolling down to another turning point, without passing through any zeros of $\mathcal{C}$. I will now examine the cosmological evolution near the special points mentioned above in more detail.

Suppose that $d\mathcal{C}/d\phi = 0$ for a finite value of ϕ and that $\mathcal{C}$ is non-singular there. Without loss of generality we can assume this occurs at $\phi = 0$ in which case (to leading order)

$$\mathcal{C} = 3nA\kappa^{-2} + B\phi^n \ , \tag{4.2}$$

where A and B are constants and $n \geq 2$. To leading order eq. (3.8) implies

$$r \sim \begin{cases} \exp\left(\frac{A}{B(n-2)}\frac{1}{\phi^{n-2}}\right) & n > 2 \\ \\ \phi^{-(A/B)} & n = 2 \end{cases} \tag{4.3}$$

Thus if $A/B > 0$, $r \to \infty$ as $\phi \to 0$, while if $A/B < 0$, $r \to 0$ instead.

Substituting these approximations into the equation for U (3.9), gives

$$U = -\frac{\kappa^4}{36}\alpha\mathcal{C}^2 + \frac{\mathcal{C}}{r^4} + \frac{1}{r^4}\begin{cases} O\left((\ln r)^{-\frac{n}{n-2}}\right) & n > 2 \\ O\left(r^{-2B/A}\right) & n = 2 \end{cases} \tag{4.4}$$

where α is an arbitrary integration constant.

Combining these expressions with the symmetric bulk Friedmann equation (3.11) gives the late time (large a and r) evolution

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\kappa^4}{36}\left(\lambda^2 - \alpha + 2\lambda\rho + \rho^2\right)\mathcal{C}^2 + \frac{\mathcal{C}}{a^4} + o\left(\frac{1}{a^4}\right) \tag{4.5}$$

If we want a non-vanishing effective gravitational constant and a vanishing effective cosmological constant, we require $\mathcal{C}(a \to \infty) \neq 0$ and $\alpha = \lambda^2$. The above equation then tends to the usual brane cosmology Friedmann equation, including a so-called dark radiation term. Disappointingly there are no quintessence-like ($1/r^c$ with $c < 2$) or dark matter-like ($1/r^3$) contributions, so this type of solution cannot explain the observed acceleration of the universe.

If instead $A = 0$ in eq. (4.2), then we are near a zero of $\mathcal{C}$. In this case

$$r \sim \exp\left(-\frac{\kappa^2\phi^2}{6n}\right) \tag{4.6}$$

for any $n > 0$, so $\phi = 0$ is a local maximum of r.

Eq. (3.9) then implies

$$\frac{U}{\mathcal{C}^2} \sim B^{-1}\begin{cases} -\phi^{2-n} & n > 2 \\ \ln|\phi| & n = 2 \end{cases} \tag{4.7}$$

Note that in order for the point $\phi = 0$ to be reachable at all, B must be negative.

Combining the Friedmann equation (3.11) with eq. (3.8) gives the evolution of ϕ with respect to time

$$\dot{\phi}^2 = 4\left(\frac{36}{\kappa^4}\frac{U}{\mathcal{C}^2} + (\lambda+\rho)^2\right)\left(\frac{d\mathcal{C}}{d\phi}\right)^2 \quad . \tag{4.8}$$

For $n \geq 2$ the dominant contribution to the first factor on the right-hand side of eq. (4.8) comes from $U/\mathcal{C}^2$ (4.7). Using this approximation, integration of the above expression reveals that $t \to \infty$ as $\phi \to 0$. Thus it takes an infinite amount of time to reach the zero of $\mathcal{C}$. As it is approached, r tends to a constant.

If $n = 1$ the dominant part of the right-hand side of eq. (4.8) is generally a constant. ϕ thus passes through the zero of $\mathcal{C}$ in finite time. As ϕ passes this point, the universe stops expanding and begins to collapse. This case is not the same as the re-collapse near the start of this section, since $\dot{\phi}$ no longer becomes zero, and the subsequent evolution of the universe is not just the evolution up to $\dot{a} = 0$ in reverse.

As $\mathcal{C} \to 0$ (for any n) the effective gravitation constant ($8\pi G = \mathcal{C}^2\kappa^4\lambda/6$) also tends to zero. Clearly this type of solution is not compatible with the standard cosmology. The solution (3.6,3.7), which corresponds to an exponential potential for $k = 0$, also has $\mathcal{C} \to 0$, although in this case that occurs as $r \to \infty$.

Using eqs. (3.8–3.10) and the components (2.8,2.9) of the Einstein tensor can be written (for any k) as $G^T_T = -6(U + rU'/4)$ and $G^r_r = -6(U - \mathcal{C}/r^4)$. If we require these to be non-singular, U and $\mathcal{C}$ must be finite for $r > 0$. As $r \to 0$ we need $\mathcal{C} \to \text{const.} + O(r^4)$ and $U = \mathcal{C}/r^4 + O(1)$. Note that $\mathcal{C}$ cannot tend to zero at $r = 0$ since eq. (3.8) implies $\mathcal{C}'/\mathcal{C} \leq 0$, and so $\mathcal{C}$ tends to a constant or diverges as $r \to 0$. However the point $r = 0$ is still singular, since this behaviour of $\mathcal{C}$ and U implies $R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho} \sim \mathcal{C}^2/r^8$. Thus there is always a curvature singularity at $r = 0$ unless $\mathcal{C} \equiv 0$. Also, because $h \sim 1/r^2$ the singularity is a finite proper distance away from the brane. The only way to avoid having a naked bulk singularity which is visible from the brane is to hide it behind an event horizon. At late time (large a, r) we want U to be negative, so if there is to be an event horizon, $\mathcal{C}$ must be chosen so that U is positive at $r = 0$. If $\mathcal{C}$ tends to a constant as $r \to 0$, it needs to be positive to give a suitable black hole solution.

Thus the only viable cosmological solutions with $\mathcal{C} \neq 0$ have bulk black holes. Our universe then starts at the initial space-like singularity [17], which is either an infinity or a maximum of $|\mathcal{C}|$. If our universe is to expand forever, it must not pass through any points for which the Friedmann equation (2.28) gives $\dot{a} = 0$. As the scale factor approaches infinity, the value of ϕ on our brane approaches a minimum of $|\mathcal{C}|$. If there are points with $\dot{a} = 0$, the universe will either take an infinite time to reach them, or when they are reached the universe will start to re-collapse. Such points exist if $\mathcal{C}$ has a zero, or if the effective cosmological constant term of the Friedmann equation happens to cancel the other terms.

5. Conformally Flat Solutions

The above rearrangement of the field equations do not work if $\mathcal{C} \equiv 0$. This is only possible if $\phi' = 0$, in which case the solution (3.5) applies, or if $k = 0$. In the second case the definition of $\mathcal{C}$ (3.1) implies that $f^2 h = r^2$, and spacetime is conformally flat. It is useful to set $U = -W^2$, in which case the field equations imply

$$-\frac{\kappa^2}{3} r\phi' = \frac{1}{W}\frac{dW}{d\phi} \ . \tag{5.1}$$

$$V(\phi) = \frac{9}{2\kappa^4}\left(\frac{dW}{d\phi}\right)^2 - \frac{6}{\kappa^2}W^2 \ , \tag{5.2}$$

The form of V in this case resembles that of supergravity theories, with W playing the role of the superpotential. The cosmology of this type of model with an exponential W has been considered elsewhere [7].

When $\mathcal{C} \equiv 0$ the Friedmann equation (3.11) becomes

$$\left(\frac{\dot{a}}{a}\right)^2 = \left\{-1 + \frac{\kappa^4}{36}\left(\lambda^2 + 2\lambda\rho + \rho^2\right)\right\} W^2 \tag{5.3}$$

which is similar to eq. (4.5), but without the 'dark radiation' term.

The equation determining $\phi(r)$ (5.1) in this case is also very similar to the corresponding $\mathcal{C} \neq 0$ equation (3.8). Again the start ($r = a = 0$) of our universe occurs at a maximum or infinity of $|W|$. The field ϕ then rolls down $|W|$ until it reaches either a minimum, in which case $a \to \infty$, or a point with $\dot{a} = 0$. If $\dot{a} \to 0$ the universe will either take an infinite amount of time to reach $\dot{a} = 0$, or the universe will start to re-collapse after passing this point. The analysis is virtually identical to that of the previous section. Again we cannot have quintessence-like terms at late time if we require a zero effective cosmological constant and a non-vanishing gravitational coupling.

The implications of singularities are different to the $\mathcal{C} \neq 0$ case. Since the metric is conformally flat it is no longer possible to hide singularities behind an event horizon. A non-singular energy-momentum tensor implies that W must be finite for all values of r. This implies that $h \sim r^2$ for small r, and so the point $r = 0$ is always an infinite proper distance away from the brane. Thus there are no black holes or naked singularities in this case.

6. A Cosmologically Viable Example

The results of the previous two sections suggest that we would like a model with a $\mathcal{C}$ or W that has a maximum and a minimum and no zeros between them. In this section I will describe one such model, whose analytic form allows the equations (3.8–3.10) and (5.1,5.2) to be integrated relatively painlessly. For convenience we will change variables to $\varphi = \kappa\phi/\sqrt{3}$. If

$$\mathcal{C} = A_1 r_*^4 \left(n(n+1) - \varphi^2\right) e^{\varphi^2/n} \ , \tag{6.1}$$

then eq. (3.8) implies

$$\frac{r^4}{r_*^4} = \left(\frac{n^2}{\varphi^2} - 1\right)\frac{1}{\varphi^{2n}} \ . \tag{6.2}$$

Thus $r = 0$ at $\varphi = n$, which is a maximum of $\mathcal{C}$, and $r \to \infty$ as $\phi \to 0$, where $\mathcal{C}$ has a minimum. Eq. (3.9) implies

$$\begin{aligned} U = A_1 \left(n(1+n) - \varphi^2\right)^2 e^{\varphi^2/n} \left\{ n^n e^{\varphi^2/n} \Gamma\left(n, \frac{\varphi^2}{n}\right) + \frac{n\varphi^{2n}}{n^2 - \varphi^2} \right\} \\ - A_2 n^n \Gamma(n) \left(n(1+n) - \varphi^2\right)^2 e^{2\varphi^2/n} \ . \end{aligned} \tag{6.3}$$

A_1, A_2 and r_* and are arbitrary constants. Here Γ is the incomplete gamma function. For positive integer values of n, $e^z \Gamma(n, z) = \Gamma(n) \sum_{k=0}^{n-1} z^k/k!$, and so the first term of (6.3) reduces to $e^{\varphi^2/n}$ times a polynomial.

To get a vanishing late time effective cosmological constant in the Friedmann equation (2.28) we need $A_2 > A_1$ and $\lambda^2 = (A_2 - A_1) n^n \Gamma(n)/(A_1 r_*^4)^2$. In order to have a black hole solution as opposed to a naked singularity, we also want U to be positive as $r \to 0$, and so we need $A_1 > 0$. The quantity $\mathcal{C}$ is then positive for all values of r.

The form of the potential can be obtained from eq. (3.10). If n is an integer it takes the form of a $2n$-th order polynomial multiplied by $e^{\varphi^2/n}$, plus a sixth order polynomial times $e^{2\varphi^2/n}$. For example if $n = 1$

$$\kappa^2 V = 6A_1(\varphi^2 - 4)e^{\varphi^2} - 6A_2(\varphi^3 + \varphi^2 - \varphi - 2)(\varphi^3 - \varphi^2 - \varphi + 2)e^{2\varphi^2} \tag{6.4}$$

This would be bounded from below if $A_2 \leq 0$, but as is mentioned above, we need A_2 positive to avoid singularity problems.

The corresponding $\mathcal{C} = 0$ solution has W proportional to eq. (6.1), and V is given by eq. (6.4) with $A_1 = 0$ and $A_2 < 0$. The evolution of r is still given by eq. (6.2). There is no black hole in this case, but no naked singularity either.

7. Summary

To fully understand the behaviour of brane cosmologies with bulk scalar fields it is not enough to just consider the fields on the brane. A full bulk solution is required. This is not generally possible, although solutions in the simpler case of a static bulk with a brane moving through it can be found. For a Z_2 symmetric brane world with a static bulk, consistency of the solution imposes restrictions on the type of theory that can exist on the brane. We find that in general the standard energy momentum condition cannot apply. However reasonable models which do give the required non-standard conservation law do exist.

We find from the bulk field equations that the variation of the scalar field (ϕ) is directly related to a projection of the bulk Weyl tensor ($\mathcal{C}/r^4$). The Friedmann equation and the bulk potential can also be determined from $\mathcal{C}$. The cosmology of the model is thus determined by the form of the function $\mathcal{C}$ (and the type of matter present on the brane).

We find that, in general, the cosmology on the brane occurs between two points with $d\mathcal{C}/d\phi = 0$ or $|\mathcal{C}| = \infty$. The universe starts ($a = 0$) at a maximum or infinity of $|\mathcal{C}|$. There are then two possibilities for the subsequent evolution. Either it can continue to expand as ϕ rolls towards a minimum of $|\mathcal{C}|$ ($a \to \infty$), or ϕ can pass through a zero of $\mathcal{C}$, at which point the universe reaches its maximum size. After this it contracts until another maximum or infinity of $|\mathcal{C}|$ is reached, where $a = 0$ again. It is also possible for the evolution to end at a point with $\mathcal{C} = 0$, in which case the scale factor a will asymptotically approach a finite value.

All the above solutions have singularities at $r = 0$, although the spacetime can have a black hole-like structure, in which case the singularity is concealed by an event horizon. If the projected bulk Weyl tensor is zero, the system requires a slightly different analysis, although the possible cosmologies are very similar. The main difference is that black hole solutions are no longer possible. However the point $r = 0$ is no longer automatically singular.

The unusual effective energy-momentum conservation equation of these models, which is required for a consistent solution, implies that the effective gravitation coupling is proportional to the function $\mathcal{C}$. Thus solutions which have $\mathcal{C}$ approaching zero are unlikely to be compatible with the standard cosmology. Requiring a non-vanishing gravitational coupling has other implications. If we also require an effective cosmological constant which vanishes at late time, then the form of the solution implies that there will be no quintessence-like terms in the Friedmann equation arising from bulk effects.

The only models which agree with the standard cosmology at late times have ϕ rolling down to a minimum of $|\mathcal{C}|$. The late time behaviour of all such solutions closely resembles that of the well known brane world cosmologies which do not have scalar fields. Thus if we keep the bulk static and Z_2 symmetric, then the introduction of scalar fields will not help resolve late time cosmological problems, such as the observed but unexplained acceleration of the universe.

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